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Correlation functions of disorder fields and parafermionic currents in Z_N Ising models

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Abstract

We study the correlation functions of parafermionic currents and disorder fields in the Z_N symmetric conformal field theory perturbed by the first thermal operator. Following the ideas of Al Zamolodchikov (Zamolodchikov 1991 *Nucl. Phys. B* **348** 619, Zamolodchikov 1990 *Nucl. Phys. B* **342** 695 and Zamolodchikov 1995 *Int. J. Mod. Phys. A* **10** 1125), we develop for the correlation functions the conformal perturbation theory at small scales and the form factors' spectral decomposition at large ones. For all N , there is an agreement between the data at the intermediate distances. We consider the problems arising in the description of the space of scaling fields in perturbed models, such as null vector relations, equations of motion and a consistent treatment of fields related by a resonance condition.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The calculation of correlation functions is one of the most interesting problems in the two-dimensional integrable quantum field theory. Complete exact solutions of this problem were found for free field models, for example the Ising model, and for the conformal field theories (CFTs).

In massive integrable field theories, an elegant way of studying the behavior of two-point correlation functions was proposed by Zamolodchikov in [1]. In the case of the perturbed Lee–Yang model, he studied short- and long-distance asymptotics of the two-point correlation function of primary fields. The short-distance behavior of the correlation function was investigated by developing the infrared (IR) safe perturbation theory [1], based on the knowledge of the exact vacuum expectation values (VEVs) of local fields found by non-perturbative methods [2–4]. The correlators in the infrared region were given by using a form factor spectral decomposition [5, 6]. A very good agreement between the asymptotics at

the intermediate distances $Mr \sim 1$ allowed us to claim that correlation functions in the Lee–Yang model are effectively described in this approach at all distances. Further development has shown that the method proposed in [1] is a simple and effective tool for an analysis of the basic properties of correlation functions in different integrable massive models.

In this paper, we continue our study [7, 8] of scaling fields in parafermionic CFT [9, 10] with the central charge

$$c = 2(N - 1)/(N + 2), \quad N = 2, 3, \dots, \quad (1)$$

perturbed by the first thermal operator ε_1 :

$$\mathcal{A} = \mathcal{A}_{\text{CFT}} + \lambda \int d^2x \varepsilon_1(x). \quad (2)$$

The resulting theory is integrable and Z_N symmetric. Depending on the sign of λ , the system is in the ordered or the disordered phase. We fix the $\lambda > 0$ phase where the \tilde{Z}_N symmetry is destroyed and vacuum expectation values of disorder operators are non-zero. In this scaling model [11, 12], we study, within the approach of [1], the correlation functions of disorder operators and parafermionic currents. These objects, as well as correlators of some W -algebra descendants, are not so easy for a direct investigation mainly because of the resonances, which appear in the construction. Namely, following the procedure of [1], we need to study the situations where the scaling dimensions D_a and D_b of some fields \mathcal{O}_a and \mathcal{O}_b satisfy the condition

$$D_a = D_b + 2n(1 - \Delta_{\varepsilon_1}), \quad n > 0. \quad (3)$$

In this case, the field \mathcal{O}_a has n th-order resonance with the field \mathcal{O}_b and there is an ambiguity in defining the renormalized field \mathcal{O}_a :

$$\mathcal{O}_a \rightarrow \mathcal{O}_a + \text{const } \lambda^n \mathcal{O}_b. \quad (4)$$

This typically results in a logarithmic scaling of the field \mathcal{O}_a [4]. We observed that according to our general formulae, the form factors of fields satisfying (3) formally coincide as functions of rapidities. This creates a problem in defining the form factors of such scaling fields, since it is expected from the general settings [13, 14] that there is a one-to-one correspondence between conformal and scaling fields. We propose to discuss a general prescription for the form factors of the fields possessing resonances in a separate publication. In this paper, we consider some examples of the phenomena.

On the other hand, by studying the short-distance behavior of the correlators, we found that the terms in the perturbative expansion, which correspond to fields with condition (3), formally diverge for integer parameters N . To proceed further with these cases, we use the fact that our exact expressions are defined for the models with arbitrary number N . We provide an analytic continuation over this parameter and obtain finite results for the correlators. We check that our prescription leads to correct expressions for the Ising model ($N = 2$ case) [15], as well as for other known cases.

Similarity between the long- and short-distance asymptotics, which we found for all N , can also be considered as an additional confirmation, supporting the consistency of the proposed form factors and the expressions for the short-distance expansions, for VEVs and for the normalizations of the scaling fields.

We choose to study the Z_N invariant Ising model, since it has several nice properties. It is related to a statistical system, which has a simple and clear description [9, 16] and many important physical applications, while the operators of this lattice model, in general, have an interesting quasi-locality property [17, 18] in the scaling limit. We recall the notions of the lattice Z_N model in section 2. In the critical points, the lattice model is described by the

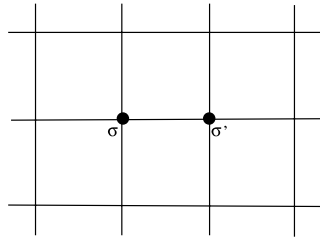


Figure 1. Z_N Ising model at the lattice.

parafermionic CFT with the central charge (1). In section 3, we introduce necessary definitions and collect basic facts about this CFT, concentrating attention on the algebraic structure of the space of conformal fields. This information is essential for further analysis of the space of form factors of local fields in the case of a massive integrable model, which we consider in section 4 as the model of QFT, defined as scattering theory with the simple Z_N symmetric S matrix [12]. We discuss an algebraic prescription for the form factors of the scaling fields. In particular, we consider the actions of deformed parafermionic currents in the space of form factors and study related questions, such as deformation of the quantum equations of motion, null vector relations and a prescription for form factors of fields, satisfying the resonance condition (3). Finally, the correlation functions of the scaling fields of the massive integrable model (2) are discussed in section 5. We concentrate our attention on the conformal perturbation theory and demonstrate how to apply the quantum equation of motion to the computing coefficients of the perturbation theory. We discuss the regularization prescription for the resonances, appearing in the perturbation theory, and also provide the results of numerical computations.

2. Lattice Z_N -Ising model

In this section, following [9] we recall the basic definitions of the two-dimensional lattice model, generalizing the well-known Z_2 -symmetric Ising model to the case with the Z_N , $N = 2, 3, \dots$, symmetry.

We consider the model of the statistical mechanics defined at the square lattice. Let spin variables σ be associated with the sites of the lattice and take values in the group Z_N , i.e. $\sigma = \omega^k$ ($k = 0, \dots, N - 1$), where $\omega = \exp\left(\frac{2\pi i}{N}\right)$. We study the local theory where the Boltzmann weight $e^{-\mathcal{H}(\sigma, \sigma')} = \mathcal{W}(\sigma, \sigma')$ depends on spins σ and σ' situated at neighboring sites, see figure 1. The partition sum of the model is, by definition,

$$\mathcal{Z} = \sum_{\text{spins}} \prod_{\text{edges}} \mathcal{W}(\sigma, \sigma'). \tag{5}$$

In the Z_N Ising model, the Boltzmann weights satisfy the Z_N symmetry $\mathcal{W}(\sigma, \sigma') = \mathcal{W}(\omega\sigma, \omega\sigma')$ and the reality condition $\mathcal{W}(\sigma, \sigma') = \mathcal{W}(\sigma^\dagger, \sigma'^\dagger)$. It means that, up to a normalization constant, the function $\mathcal{W}(\sigma, \sigma')$ has the form

$$\mathcal{W}(\sigma, \sigma') = \sum_{k=0}^{N-1} \mathcal{W}_k (\sigma^\dagger \sigma')^k, \quad \mathcal{W}_0 = 1, \tag{6}$$

where the parameters \mathcal{W}_k of the model are real non-negative numbers satisfying the equation $\mathcal{W}_k = \mathcal{W}_{N-k}$.

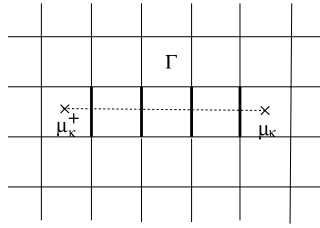


Figure 2. Two-point correlator of disorder operators μ_k and μ_k^\dagger .

Important information about the model is encoded in a set of its correlation functions. For example, the correlation functions of the spin operators are defined at the lattice as

$$\langle \sigma_{k_1}(x_1) \cdots \sigma_{k_s}(x_s) \rangle = \frac{1}{Z} \sum_{\text{spins}} \sigma^{k_1}(x_1) \cdots \sigma^{k_s}(x_s) \prod_{\text{edges}} \mathcal{W}(\sigma, \sigma').$$

We will study a system in the thermodynamic limit, assuming appropriate periodic conditions at infinity. The one-point correlation functions $\langle \sigma_k \rangle$ serve as a measure of order in the system. Other important lattice operators are the disorder operators [10, 17, 18]. Consider a directed path Γ going through the points of a dual lattice and intersecting the bonds of the original lattice. Let all weights \mathcal{W}_s at the bonds, crossed by path Γ , be changed in order to become $\tilde{\mathcal{W}}_s^{(k)} = \mathcal{W}_s \omega^{sk}$. The presence of a dislocation along the path introduces in the system a fractional domain wall that favors a discontinuity in the value of the neighboring spins by k . The partition function \tilde{Z}_Γ on the inhomogeneous lattice now differs from the original one. We interpret the dislocation as an insertion of two operators μ_k and $\mu_k^\dagger = \mu_{N-k}$ situated at the sites of the dual lattice as is shown in figure 2.

By definition, the two-point correlation function of conjugate disorder operators is given as

$$\langle \mu_k(x_1) \mu_k^\dagger(x_2) \rangle_\Gamma = \frac{1}{Z} \tilde{Z}_\Gamma.$$

Here, $x_{1,2}$ are the coordinates of the corresponding operators. This interpretation turns out to be useful since the dependence on the contour Γ is not very strong. As in the Ising model [17], one can make contour deformations by closed paths, without changing the correlation function. With the freedom of making these deformations, we use the prescription of attaching contours from infinity to each of the points x_j of the dual lattice, as is shown in figure 3. Now, the definition of the two-point correlation functions can be immediately extended to the multi-point case, including disorder operators with a total non-zero Z_N charge.

In a general case of correlators of spin and disorder operators, only an absolute value of the correlation function remains to be path independent, and one should fix the relative position of contours [10]. This is because the correlation functions of the product of spin and disorder operators, before and after a complete counterclockwise rotation of the disorder variable around the order variable, differ by a phase ω^{-kl} . Equivalently, this happens whenever a μ_k -path crosses a σ_l variable. Operators with these properties are called mutually quasi-local with the exponent $\gamma_{kl} = -kl/N$.

Order and disorder parameters are basic operators in the theory. Other operators are constructed as their operator products. In general, if the distance $|x - y|$ between the operators is much smaller than the correlation length, then the local operators $\psi_{l,k}$, appearing at the operator product

$$\sigma_l(x) \mu_k(y) = C_{lk}(x - y) \psi_{l,k}(x) + \cdots,$$

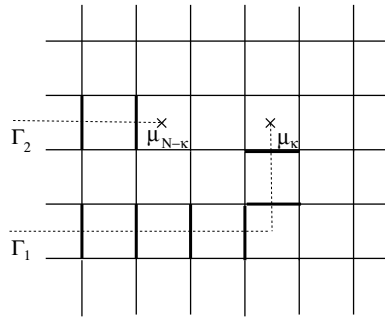


Figure 3. Correlation function $\langle \mu_k \mu_{N-k} \rangle$.

do obey the parastatistics [18]. In particular, we will further study the correlation functions of two parafermionic currents $\psi = \psi_{1,1}$ and ψ^\dagger in the scaling limit.

Under the Kramers Wannier duality [17], order parameters become disorder parameters and vice versa. The hyperplane of self-duality has the dimension $[\frac{N}{4}]$. For the Ising model ($N = 2$ case) and three-state Potts model ($N = 3$ case), the system has a second-order phase transition at the self-dual point. For $N = 4$, it coincides with the well-known Ashkin–Teller model. For prime $N \geq 5$, the Z_N theory has, as a rule, three phases (ordered, disordered and Kosterlitz–Thouless phases). For non-prime N , the phase structure is more complicated. As an example, we consider the phase diagram of the Z_5 model [16], see figure 4.

Here phase I, the ordered phase, is characterized by the conditions $\langle \sigma_l \rangle \neq 0$ and $\langle \mu_k \rangle = 0$. In the disordered phase II, the situation is inverse: $\langle \sigma_l \rangle = 0$ and $\langle \mu_k \rangle \neq 0$. Finally, in the Kosterlitz–Thouless phase III, the expectation values of the operators of both types are zero $\langle \sigma_l \rangle = \langle \mu_k \rangle = 0$. The line FB denotes the self-duality region. It contains two symmetrically situated ‘bifurcation points’ **C** and **C***. Along the line **CC***, the model has a first-order phase transition and ordered and disordered phases may coexist. The points **C** and **C*** are integrable and critical [16, 28]. The theory in these points has $Z_N \times \tilde{Z}_N$ symmetry and is described by the continuous parafermionic CFT constructed in [9]. In the scaling limit in the vicinity of a critical point the order σ_l and disorder μ_k operators, as well as parafermions ψ , are described by the fields, depending on the continuous space parameters. We preserve for these fields the same notations as on the lattice.

Due to the conformal invariance and infinite-dimensional symmetry of the critical theory [19], the analysis of its correlation functions, as well as the structure of its space of states, simplifies drastically. This was described in details in [9, 10]. Based on CFT results, one can study the basic properties of the correlation functions of the Z_N models in the vicinity of the critical point by application of the conformal perturbation theory. Again, the simplest perturbations are those which are integrable due to the presence of an infinite set of integrals of motion [13]. Different integrable perturbations of the conformal field theories [9] were studied in [11].

In this paper, we study a vicinity of the critical point **C** in phase II, where temperature deformation leads to the appearance of the finite correlation length and to the non-zero vacuum expectation value of the disorder parameters [11, 12, 20]. In the QFT language, this is the massive perturbation of parafermionic CFT (2) by the most relevant first thermal operator ε_1 , which destroys the dual \tilde{Z}_N symmetry and preserves the symmetry Z_N .

One of the questions which we would like to address in our study is a structure of the space of scaling fields. We expect that the space of composite quasi-local fields constructed

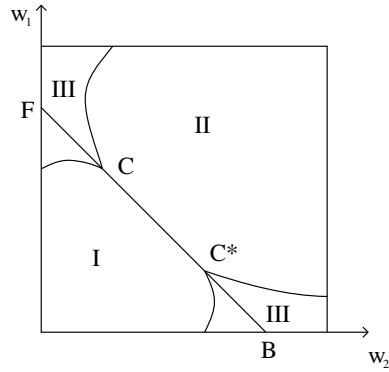


Figure 4. The phase diagram of the Z_5 model.

from operators σ_k and μ_l in the vicinity of the critical point will be essentially the same, as in the CFT. Let us note that this statement was supported by several results on counting of local operators in various integrable models [14, 22]. To understand this problem, we tried to apply in [7, 8] the knowledge of the algebraic structure of lattice operators, which is based on a deformation of conformal algebras [23–26]. According to this idea, we first recall the structure of quasi-local fields in the CFT point. Then, we will try to apply the clear and simple algebraic scheme for the investigations of matrix elements of scaling fields on the basis of asymptotic states and also for studying their correlation functions off-criticality.

3. Space of states in parafermionic CFT

In this section, we recall basic facts [9, 10] about the conformal field theory with the parafermionic symmetry which describes critical points of the Z_N Ising model [9] and related models [32].

3.1. Parafermionic symmetry

In the conformal limit the order σ_k and disorder μ_k parameters, which determine long-range correlations of spins and dual spins, have the anomalous dimensions

$$2d_k = \frac{k(N - k)}{N(N + 2)}. \tag{7}$$

Their Z_N and dual \tilde{Z}_N charges, respectively, are equal to k . Under the action of the Z_N symmetry, spin fields transform as

$$\sigma_k \rightarrow \omega^{kn} \sigma_k, \quad n \in Z. \tag{8}$$

The transformation law for operators μ_k under the action of the dual \tilde{Z}_N symmetry looks similar:

$$\mu_l \rightarrow \omega^{ln'} \mu_l, \quad n' \in Z. \tag{9}$$

The spin and disorder fields are the basic operators in the theory. All other fields are constructed from them. The composite fields are naturally separated into families with the fixed additive $Z_N \times \tilde{Z}_N$ charges (k, l) . The members of the family behave under the $Z_N \times \tilde{Z}_N$ transformation as

$$\Phi \rightarrow \omega^{kn+ln'} \Phi, \quad n, n' \in Z.$$

The parafermionic fields ψ_k and $\bar{\psi}_k$ generalizing the usual Ising model fermions appear in the OPE of the order and disorder fields:

$$\begin{aligned} \sigma_k(z, \bar{z})\mu_k(0, 0) &= |z|^{-4d_k} z^{\Delta_k} [\psi_k(0) + \dots], \\ \sigma_k(z, \bar{z})\mu_k^\dagger(0, 0) &= |z|^{-4d_k} \bar{z}^{\Delta_k} [\bar{\psi}_k(0) + \dots]. \end{aligned} \tag{10}$$

These currents are holomorphic and generate the infinitely dimensional symmetry due to the conservation laws

$$\partial_{\bar{z}}\psi_k = 0, \quad \partial_z\bar{\psi}_k = 0. \tag{11}$$

We concentrate our attention on the simplest solution of the associativity condition for the operator algebra of currents, which corresponds to CFT with the central charge (1), and the conformal dimensions of currents:

$$\Delta_k = \frac{k(N - k)}{N}. \tag{12}$$

The fields $\psi = \psi_1$ (and respective anti-chiral currents) are the basic ones in the parafermionic algebra. It is convenient for us to consider, as well, conjugate currents $\psi_{N-1} = \psi^\dagger$.

In the conformal model, the space of states splits naturally into a direct sum of subspaces with the specified $Z_N \times \tilde{Z}_N$ charge (k, l) :

$$\{F\} = \oplus \{F\}_{[m, \bar{m}]}, \quad N \geq m, \bar{m} \geq 1 - N, \tag{13}$$

where $[m, \bar{m}] = [k + l, k - l]$, $m + \bar{m} \in 2Z$. In these notations, parafermionic currents and order-disorder fields belong to the following subspaces:

$$\begin{aligned} \psi &\in \{F\}_{[2, 0]}, & \psi^\dagger &\in \{F\}_{[-2, 0]}, \\ \bar{\psi} &\in \{F\}_{[0, 2]}, & \bar{\psi}^\dagger &\in \{F\}_{[0, -2]}, \\ \sigma_k &\in \{F\}_{[k, k]}, & \mu_k &\in \{F\}_{[k, -k]}. \end{aligned} \tag{14}$$

Conformal fields are classified according to the representations of the parafermionic algebra. The action of the parafermionic generators A_ν (A_ν^\dagger) is defined by the OPE

$$\begin{aligned} \psi(z)\Phi_{[m, \bar{m}]} &= \sum z^{-\frac{m}{N} + n - 1} A_{\frac{1+m}{N} - n} \Phi_{[m, \bar{m}]}, \\ \psi^\dagger(z)\Phi_{[m, \bar{m}]} &= \sum z^{\frac{m}{N} + n - 1} A_{\frac{1-m}{N} - n}^\dagger \Phi_{[m, \bar{m}]}. \end{aligned} \tag{15}$$

Note that if $\Phi_{[m, \bar{m}]} \in F_{[m, \bar{m}]}$ has the conformal dimensions (d, \bar{d}) , then the conformal dimensions of fields

$$A_\nu \Phi_{[m, \bar{m}]} \in \{F\}_{[m+2, \bar{m}]}, \quad A_\nu^\dagger \Phi_{[m, \bar{m}]} \in \{F\}_{[m-2, \bar{m}]}$$

are $(d - \nu, \bar{d})$.

Order and disorder fields are the primaries of the parafermionic algebra. For instance, the following equations hold for $n \geq 0$:

$$A_{\frac{1+k}{N} + n} \mu_k = A_{\frac{1-k}{N} + n + 1}^\dagger \mu_k = 0, \quad \bar{A}_{\frac{1-k}{N} + n + 1} \mu_k = \bar{A}_{\frac{1+k}{N} + n}^\dagger \mu_k = 0. \tag{16}$$

All other fields of the model are obtained by the action of the currents $\psi, \bar{\psi}$ on the fields μ_k . The space of states of the CFT decomposes into a direct sum of irreducible representations of the parafermionic algebra:

$$\{F\} = \oplus_{k=0}^{N-1} [\mu_k]_{A, \bar{A}}. \tag{17}$$

This space can also be obtained by an application of parafermionic generators to the order parameters σ_k . Moreover, in some situations we will need another infinite symmetry description of parafermionic CFT, namely the W symmetry, which we recall in the following subsection.

3.2. *W algebra symmetry*

The space of fields in parafermionic CFT allows a classification with respect to another infinite dimensional symmetry algebra, the so-called W_N algebra [27]. The generators of the latter $W_2(z), W_3(z), \dots$ appear at the operator product of parafermionic currents:

$$\psi(z)\psi^\dagger(0) = \frac{1}{z^{2\Delta_1}} \left(1 + z^2 \frac{N+2}{N} W_2(0) + z^3 \left(\frac{1}{N^{\frac{3}{2}}} W_3(0) + \frac{N+2}{2N} \partial W_2(0) \right) + \dots \right). \quad (18)$$

Here, W_2 currents with spin-2 generate Virasoro algebra with the central charge being given by equation (1). The currents W_3 have spin-3. They generate the whole algebra, including the higher spin currents, which are omitted in equation (18).

Currents of W algebra, and respective anti-chiral currents, have zeroth Z_N charges. Acting on the highest weight fields, they create an irreducible representation. From the viewpoint of the $W_N \times \bar{W}_N$ symmetry, each of the spaces $[\mu_k]_{A, \bar{A}}$ expands into a direct sum of representations. Namely, let us denote as $(\psi^\dagger)^l \mu_k$ the field with the minimal conformal dimension, which can be obtained by the l -times application of the parafermionic generators ψ^\dagger to μ_k :

$$(\psi^\dagger)^l \mu_k = A_{\frac{2l-1-k}{N}}^\dagger A_{\frac{2l-3-k}{N}}^\dagger \dots A_{\frac{1-k}{N}}^\dagger \mu_k. \quad (19)$$

Its conformal dimensions are easily computed from (7). Then, up to a normalization, the following relations take place:

$$\begin{aligned} \Phi_{k-2l, -k+2\bar{l}}^{(k)} &= (\psi^\dagger)^l (\bar{\psi})^{\bar{l}} \mu_k, & l, \bar{l} &= 0, 1, \dots, k, \\ \Phi_{k+2l, -k-2\bar{l}}^{(k)} &= (\psi)^l (\bar{\psi}^\dagger)^{\bar{l}} \mu_k, & l, \bar{l} &= 0, 1, \dots, N-k. \end{aligned} \quad (20)$$

These fields $\Phi_{m, \bar{m}}^{(k)}$ are the W algebra primaries. The action of the generators of W algebra on it creates the irreducible representation $[\Phi_{m, \bar{m}}^{(k)}]_{W, \bar{W}}$. The explicit values of the conformal dimensions $(d_m^{(k)}, \bar{d}_m^{(k)})$ of these fields are given as follows:

$$d_m^{(k)} = \frac{(k+1)^2 - 1}{4(N+2)} - \frac{m^2}{4N}, \quad -m \leq k \leq m. \quad (21)$$

For the cases, when $|m| > k$, we use the relation $d_m^{(k)} = d_{m-N}^{(N-k)}$, which follows from the Z_N symmetry condition:

$$\Phi_{m\bar{m}}^{(k)} = \Phi_{m-N, -N+\bar{m}}^{(N-k)}. \quad (22)$$

For example, the physically important energy fields $\varepsilon_k = \Phi_{0,0}^{(2k)}$ are among these primaries. Operators ε_k are local with respect to all fields and have the conformal dimensions

$$D_k = k(k+1)/(N+2). \quad (23)$$

We refer to paper [9] for further details. Let us only comment here that the Z_N symmetric parafermionic CFT can be equivalently considered as a particular case of the conformal field theory \mathcal{WM}_N^p , introduced in [27]. Namely, for the fixed value of N it has the smallest central charge (1) among all rational unitary minimal conformal theories with the extended W_N algebra, which corresponds to the parameter $p = N + 1$.

We will use the above-represented conformal data in the following sections to describe the perturbed conformal operators, for which we will again preserve the same CFT notations.

4. Form factor approach

Now we turn to the operators in the corresponding massive integrable theory (2), which allows several equivalent descriptions. In this section, we describe it as a two-dimensional QFT model with the factorized scattering of Z_N charged particles. The particles $a \in \{1, \dots, N - 1\}$ in the Z_N ($N = 2, 3, 4, \dots$) symmetric models have masses [20]

$$M_a = M \frac{\sin(\pi a/N)}{\sin(\pi/N)}. \quad (24)$$

The antiparticle a^\dagger is, by definition, identified with the particle $N - a$. The scattering matrix of the lightest particles $a = 1$ has a simple form [12]:

$$S_{11}(\beta) = \frac{\sinh\left(\frac{\beta}{2} + \frac{i\pi}{N}\right)}{\sinh\left(\frac{\beta}{2} - \frac{i\pi}{N}\right)}. \quad (25)$$

The S matrices for higher particles are also diagonal and can be extracted from S_{11} , according to a standard bootstrap prescription. For example, the scattering matrix between particle 1 and antiparticle 1^\dagger is $S_{11^\dagger}(\beta) = S_{11}(i\pi - \beta)$.

The knowledge of the exact spectrum (24) and the scattering matrix (25) allows us to study the correlation functions of the theory, by using its spectral decomposition into a series of form factors. The form factors

$$\langle \mathcal{O}(x) | \beta_1, \dots, \beta_n \rangle_{a_1, \dots, a_n} \quad (26)$$

of the scaling field $\mathcal{O}(x)$ are matrix elements of this operator in a basis of asymptotic states, formed by particle creation operators. We assume that the particles, labeled by a_1, \dots, a_n , have rapidities β_1, \dots, β_n . Function (26) should satisfy to some analytical properties and, also, to a set of functional equations the so-called form factor axioms [5, 6], to guarantee the (quasi)locality of scaling fields.

A usual problem in the form factor approach (see, for example, [29–31] for the Z_N Ising model case) is that it is not so easy to determine which scaling field is described by the given functions, satisfying proper functional equations and analyticity conditions. Moreover, for a matching with short-distance formulae, it is necessary to determine the normalization of the scaling fields, described by form factors. In our construction [7, 8], we proposed some algebraic approach to solve these problems. Still, there are many subtle questions in this direction. We would like to discuss some of them in the following sections.

4.1. Construction of free fields for form factors

In [7, 8], we have followed the algebraic approach [23, 25, 26, 33, 34] to the form factors of the scaling limit of the ABF model [32]. This lattice model falls into the same universality class, as the Z_N Ising model. In the corner transfer matrix approach, its hidden symmetry is a deformation of CFT symmetry algebras (7)–(23). We explore algebraic maps in the space of form factors to produce exact expressions for form factors of scaling fields. Our basic prescription, derived in [7, 8] for form factors, can be reformulated in short as follows. We

introduce the notations³

$$\begin{aligned} \mathcal{B}^\dagger(\beta)_{m,\bar{m}}^{(k)} &= \frac{e^{\frac{m-\bar{m}}{2N}\beta}}{\sqrt{2\sin\frac{\pi}{N}}} \sum_{a=\pm} a e^{\frac{i\pi}{2N}(k+1-\frac{m-\bar{m}}{2})a} \mathcal{Z}_a^\dagger(\beta), \\ \mathcal{B}(\beta')_{m,\bar{m}}^{(k)} &= -\frac{e^{-\frac{m-\bar{m}}{2N}\beta'}}{\sqrt{2\sin\frac{\pi}{N}}} \sum_{b=\pm} b e^{\frac{i\pi}{2N}(-k-1-\frac{m-\bar{m}}{2})b} \mathcal{Z}_b(\beta'). \end{aligned} \quad (27)$$

The explicit expressions for the form factors of fields $(\psi^\dagger)^l (\bar{\psi})^{\bar{l}} \mu_k$ in the perturbed theory are given for $m = k - 2l, \bar{m} = k - 2\bar{l}$ as

$$\langle (\psi^\dagger)^l (\bar{\psi})^{\bar{l}} \mu_k | \{\beta\}, \{\beta'\}_{(n,n')} \rangle = C_{m,\bar{m}}^{(k)} \left\langle \left\langle \prod_1^n \mathcal{B}(\beta_j)_{m,\bar{m}}^{(k)} \prod_1^{n'} \mathcal{B}^\dagger(\beta'_j)_{m\bar{m}}^{(k)} \right\rangle \right\rangle. \quad (28)$$

We assume in this equation that the form factor is zero unless the Z_N neutrality condition $2(n' - n) = m + \bar{m}$ is satisfied. The label n stands for the number of lightest antiparticles 1^\dagger , carrying the Z_N charge -2 , and n' means the number of particles 1 with the Z_N charge 2 . The constant $C_{m,\bar{m}}^{(k)}$ is determined by the normalization of the scaling field. In what follows, we will specify that its value for spinless fields and for parafermionic currents is in agreement with the conformal normalization. In equation (28), we used a shorthand notation for the state with these numbers of particles and antiparticles:

$$|\{\beta\}, \{\beta'\}_{(n,n')}\rangle \equiv |\beta_1, \dots, \beta_n, \beta'_1, \dots, \beta'_{n'}\rangle_{1^\dagger \dots 1^\dagger, 1 \dots 1}. \quad (29)$$

The symbol of the ordered product of particle creation operators in equation (28) is used for the object

$$\prod_1^n \mathcal{B}(\beta_j) = \mathcal{B}(\beta_1) \cdots \mathcal{B}(\beta_n), \quad (30)$$

which is a linear combination of the products of exponential-free bosonic fields $\mathcal{Z}_\pm(\beta)$; see [23] for details. The expectation value of a product of operators $\mathcal{Z}_a(\beta)$ and $\mathcal{Z}_a^\dagger(\beta)$ over the Fock vacuum can be computed by applying the Wick theorem:

$$\begin{aligned} \langle \langle \mathcal{Z}_{a_1}(\beta_1) \cdots \mathcal{Z}_{a_n}(\beta_n) \mathcal{Z}_{b_1}^\dagger(\beta'_1) \cdots \mathcal{Z}_{b_m}^\dagger(\beta'_m) \rangle \rangle &= \prod_{i < j} \langle \langle \mathcal{Z}_{a_i}(\beta_i) \mathcal{Z}_{a_j}(\beta_j) \rangle \rangle \\ &\times \prod_{i < j} \langle \langle \mathcal{Z}_{b_i}^\dagger(\beta'_i) \mathcal{Z}_{b_j}^\dagger(\beta'_j) \rangle \rangle \prod_{i,j} \langle \langle \mathcal{Z}_{a_j}(\beta_j) \mathcal{Z}_{b_i}^\dagger(\beta'_i) \rangle \rangle. \end{aligned} \quad (31)$$

The contraction rules of two operators in this equation are determined in terms of the meromorphic functions $\zeta(\beta), \zeta^\dagger(\beta)$, given in appendix B, as follows (we assume that $\beta = \beta_1 - \beta_2$):

$$\begin{aligned} \langle \langle \mathcal{Z}_a(\beta_1) \mathcal{Z}_b(\beta_2) \rangle \rangle &= \langle \langle \mathcal{Z}_{-a}^\dagger(\beta_1) \mathcal{Z}_{-b}^\dagger(\beta_2) \rangle \rangle = \zeta(\beta) \frac{\sinh\left(\frac{\beta}{2} + \frac{i\pi}{2N}(a-b)\right)}{\sinh\frac{\beta}{2}}, \\ \langle \langle \mathcal{Z}_a(\beta_1) \mathcal{Z}_b^\dagger(\beta_2) \rangle \rangle &= \langle \langle \mathcal{Z}_{-a}^\dagger(\beta_1) \mathcal{Z}_{-b}(\beta_2) \rangle \rangle = \zeta^\dagger(\beta) \cosh\left(\frac{\beta}{2} - \frac{i\pi}{2N}(a+b)\right). \end{aligned} \quad (32)$$

³ Matrix elements of operators \mathcal{Z}_a and \mathcal{Z}_a^\dagger are defined by rules (30) and (31), respectively. A more explicit definition of these operators can be found in [23].

4.2. Parafermionic current actions

Let us in short comment the algebraic structures, encoded in the prescription for form factors described above. We obtained equation (28) starting from the thermal operators form factors case $m = -\bar{m}$ and $n' = n$ [23] as a result of the parafermionic current actions [7] on the bosonic operators:

$$\begin{aligned}
 \mathcal{Z}_a^\dagger(\beta) &\xrightarrow{(\psi)} \mathcal{Z}_a^\dagger(\beta) e^{\frac{\beta}{N} - \frac{i\pi}{2N} a}, & \mathcal{Z}_b(\beta) &\xrightarrow{(\psi)} \mathcal{Z}_b(\beta) e^{-\frac{\beta}{N} - \frac{i\pi}{2N} b}, \\
 \mathcal{Z}_a^\dagger(\beta) &\xrightarrow{(\bar{\psi})} \mathcal{Z}_a^\dagger(\beta) e^{-\frac{\beta}{N} + \frac{i\pi}{2N} a}, & \mathcal{Z}_b(\beta) &\xrightarrow{(\bar{\psi})} \mathcal{Z}_b(\beta) e^{\frac{\beta}{N} + \frac{i\pi}{2N} b}, \\
 \mathcal{Z}_a^\dagger(\beta) &\xrightarrow{(\psi)^\dagger} \mathcal{Z}_a^\dagger(\beta) e^{-\frac{\beta}{N} + \frac{i\pi}{2N} a}, & \mathcal{Z}_b(\beta) &\xrightarrow{(\psi)^\dagger} \mathcal{Z}_b(\beta) e^{\frac{\beta}{N} + \frac{i\pi}{2N} b}, \\
 \mathcal{Z}_a^\dagger(\beta) &\xrightarrow{(\bar{\psi})^\dagger} \mathcal{Z}_a^\dagger(\beta) e^{\frac{\beta}{N} - \frac{i\pi}{2N} a}, & \mathcal{Z}_b(\beta) &\xrightarrow{(\bar{\psi})^\dagger} \mathcal{Z}_b(\beta) e^{-\frac{\beta}{N} - \frac{i\pi}{2N} b}.
 \end{aligned} \tag{33}$$

Such a prescription determines a set of maps in the space of multiparticle form factors, for example⁴

$$\langle 0 | \Phi_{m,\bar{m}}^{(k)} | \{\beta\}, \{\beta'\} \rangle_{(n,n')} \xrightarrow{(\psi)} \langle 0 | (\psi) \Phi_{m,\bar{m}}^{(k)} | \{\beta\}, \{\beta'\} \rangle_{(n-1,n')}. \tag{34}$$

This relation and similar relations for other parafermionic current actions are understood as follows. We assume that the multiparticle form factors for the field $\Phi_{m,\bar{m}}^{(k)}$ are given as matrix elements of a linear combination of products of n operators $\mathcal{Z}_b(\beta)$ and n' operators $\mathcal{Z}_a^\dagger(\beta)$. Then, the form factors of the field $(\psi) \Phi_{m,\bar{m}}^{(k)}$ will be given by the modified bosonization prescription, where the number of antiparticles is less by one, and the change (33) is provided for each of \mathcal{Z}_a and \mathcal{Z}_a^\dagger operators. One can see that this prescription obviously agrees with equation (28). Equation (33) was obtained from the deformed parafermionic action. For this reason, the following identification was proposed for $l, \bar{l} = 0, 1, \dots, k$ in [7]:

$$\langle 0 | \Phi_{k-2l, -k+2\bar{l}}^{(k)} | \{\beta\}, \{\beta'\} \rangle_{(n,n')} = \langle 0 | (\psi^\dagger)^l (\bar{\psi})^{\bar{l}} \mu_k | \{\beta\}, \{\beta'\} \rangle_{(n,n')}. \tag{35}$$

The validity of this equation for multiparticle form factors with $\bar{m} = -m$ was checked by comparing a prescription for the form factors (27)–(32) with the form factors of the scaling fields on the left-hand side of this equation, obtained in the prescription for the deformed W algebra primaries [33]. We also get a clear evidence of the correctness of this identification by studying the correlation functions of the order and disorder fields in [7]⁵. We consider equation (35) as an off-critical analogue of equation (20).

It is possible to further reproduce the study of the structure of the space of form factors of scaling fields in an analogy with the correspondent algebraic description of CFT. We note here that together with equation (20), the form factor prescription (27)–(32) also satisfies the charge conjugation condition (22). Using the Z_N symmetry condition $\psi = \Phi_{N-2, -N}^{(N)}$, it is possible to derive the form factors of the parafermionic currents. A simple observation is that these matrix elements are related to the form factors of the field $\Phi_{2,0}^{(2)} = (\psi) \varepsilon_1$ as follows (see also [29]):

$$\langle \psi | \{\beta\}, \{\beta'\} \rangle_{(n-1,n)} = \lambda \sqrt{\frac{2}{N}} \left(\sum^{n-1} e^{-\beta_j} + \sum^n e^{-\beta'_j} \right)^{-1} \langle \Phi_{2,0}^{(2)} | \{\beta\}, \{\beta'\} \rangle_{(n-1,n)}. \tag{36}$$

⁴ We use in equation (34) the conformal notations $(\psi) \Phi_{m,\bar{m}}^{(k)}$, since this equation is a perturbed analogue of equation (19).

⁵ Moreover, we demonstrated in [7] that the deformed parafermionic currents, acting on the deformed vertex operators, also reproduce VEVs for perturbed W algebra primaries and, in particular, explain the factorized form of the resulting VEVs (56) of the fields $\Phi_{m,\bar{m}}^{(k)}$.

This equation is an off-critical analogue of the quantum equation of motion (11) for the perturbed parafermionic currents:

$$\frac{\partial}{\partial \bar{z}} \psi(z, \bar{z}) = \lambda \sqrt{\frac{2}{N}} (\psi) \epsilon_1(z, \bar{z}) = \lambda \sqrt{\frac{2}{N}} \Phi_{2,0}^{(2)}(z, \bar{z}). \quad (37)$$

In the general multiparticle case, this equation follows from the trigonometric function identities. Further, we will use equation (37) in the short-distance expansion.

Our next comment on a mapping between form factors and interpreting our form factors as matrix elements of the scaling fields is as follows. Formally applying the prescription of [7], we arrive at the fact that the null vectors equation (16) survives after the perturbation (see [22], where a similar phenomenon was studied in the sine-Gordon model context). It can be directly checked that for perturbed fields, we have relations

$$A_{\frac{1+k}{N}} \mu_k = 0, \quad \bar{A}_{\frac{1+k}{N}}^\dagger \mu_k = 0. \quad (38)$$

The first equation here follows from the parafermionic currents action procedure, introduced in [7]. Indeed, one can check that taking the $\alpha \rightarrow 0$ limit in the expression

$$\langle A_{\frac{1+k}{N}} \Phi_{k,-k}^{(k)} | \{\beta\}, \{\beta'\} \rangle_{(n-1,n)} \sim \lim_{\alpha \rightarrow \infty} e^{-\frac{1+k}{N} \alpha} \left\langle \left\langle \prod_{i=1}^{n-1} \mathcal{B}(\beta_i)_{k,-k}^{(k)} \prod_{j=1}^n \mathcal{B}^\dagger(\beta'_j)_{k,-k}^{(k)} \mathcal{B}(\alpha)_{k,-k}^{(k)} \right\rangle \right\rangle, \quad (39)$$

we get exact zero for an arbitrary particle number n . For the second equation, after the application of the Wick theorem, we find that the condition

$$\langle 0 | \bar{A}_{\frac{1+k}{N}}^\dagger \mu_k | \beta_1, \dots, \beta_{s+1}, \beta'_1, \dots, \beta'_s \rangle = 0, \quad (40)$$

for the arbitrary $2s + 1$ particle matrix element of the scaling field $\bar{A}_{\frac{1+k}{N}}^\dagger \mu_k$, is reduced to proving the following trigonometric identity:

$$\begin{aligned} & \sum_{\{a_j\}, \{b_j\}} \prod_{i=1}^{s+1} a_i \prod_{j=1}^s b_j e^{-\frac{i\pi}{N} (k+1) b_j} \prod_{i=1}^{s+1} \prod_{j=1}^s \cosh \left(\frac{\beta_i - \beta'_j}{2} - \frac{i\pi}{2N} (a_i + b_j) \right) \\ & \times \prod_{i < j}^{s+1} \sinh \left(\frac{\beta_i - \beta_j}{2} + \frac{i\pi}{2N} (a_i - a_j) \right) \prod_{i < j}^s \sinh \left(\frac{\beta'_i - \beta'_j}{2} - \frac{i\pi}{2N} (b_i - b_j) \right) = 0, \end{aligned}$$

which follows from equation (28).

Still, the null vector conditions, as well as other relations in the space of form factors of scaling fields, have to be studied in depth, due to a possible appearance of the fields, satisfying conditions (3) and (4). The following example illustrates the fact that we have to be very careful with the analysis of the structure of the off-critical fields. We consider the multiparticle form factors of the perturbed fields:

$$\psi_k \bar{\psi}_k^\dagger = \Phi_{2k-N, N-2k}^{(N)}.$$

The expressions for these form factors, computed by a direct application of equation (28), up to a normalization, coincide, as functions of rapidities, with the corresponding results for the field $\Phi_{2k,-2k}^{(2)} = \Phi_{2k-N, N-2k}^{(N-2)}$. The equality follows from the identities for trigonometric functions. In this way, after the perturbation, we get, formally, that two fields have the same form factors, while they are different at the criticality. This disagrees with the statement that conformal and massive scaling fields should be in one-to-one correspondence [13]. The point is that this formal coincidenceness of expressions for different fields happens in a very specific case, in which the form factor prescription has to be worked out more carefully. Using the

explicit values for the conformal dimensions (21), we see that the following relations between the scaling dimensions of these fields take place:

$$2d_{2k}^{(0)} - 2d_{2k}^{(2)} = (2 - 2D_1). \tag{41}$$

Since $D_1 = \Delta_{\varepsilon_1}$ is the conformal dimension of the thermal operator ε_1 , this equation is exactly condition (3) and the fields $\psi_k \bar{\psi}_k^\dagger$ are in the first-order resonance⁶ with fields $\Phi_{2k,-2k}^{(2)}$, i.e. there is an ambiguity (4) in its definition:

$$\psi_k \bar{\psi}_k^\dagger \sim \psi_k \bar{\psi}_k^\dagger + \text{const } \lambda \Phi_{2k,-2k}^{(2)}. \tag{42}$$

Moreover, the spinless fields $\psi_k \bar{\psi}_k^\dagger$ formally have divergent expectation values; therefore, we need to work out a regularized prescription for its normalized multiparticle matrix elements.

To introduce a well-defined form factor of the field $\psi_k \bar{\psi}_k^\dagger$, we propose to use the freedom in choosing the index N . Namely, let us provide an analytic continuation of the expression for the multi-point form factor of this field, by changing $N \rightarrow N + \epsilon$, where the regularization parameter ϵ is a small number, which will be set to zero in the end. After the change, the vacuum expectation value becomes finite and proportional to $1/\epsilon$. We can divide the ϵ -deformed expressions, obtained from equation (28), into the correspondent finite value of the VEV and normalize the null point form factors to 1. The same analytic continuation and normalization are provided for the field $\Phi_{2k,-2k}^{(2)}$. Then, we propose to define the regularized form factors of $\psi_k \bar{\psi}_k^\dagger$ as a result of the formal calculation:

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\frac{\langle \psi_k \bar{\psi}_k^\dagger | \{\beta\}, \{\beta'\} \rangle_{(n,n)}^{(\epsilon)}}{\langle \psi_k \bar{\psi}_k^\dagger \rangle^{(\epsilon)}} - \frac{\langle \Phi_{2k,-2k}^{(2)} | \{\beta\}, \{\beta'\} \rangle_{(n,n)}^{(\epsilon)}}{\langle \Phi_{2k,-2k}^{(2)} \rangle^{(\epsilon)}} \right). \tag{43}$$

This is one of the possible prescriptions in fixing the ambiguity (4). In the ultraviolet (UV) regime, we should provide an analogous regularization for the field $\psi_k \bar{\psi}_k^\dagger$, which leads to vanishing expectation values of regularized fields.

To support this construction, we applied the regularization procedure for finding the form factors of the descendant field $E_1 = W_{-1}^{(3)} \bar{W}_{-1}^{(3)} \varepsilon_1$ in the $N = 4$ model⁷. The field E_1 in this theory satisfies the first-order resonance condition with the identity operator. For form factors of this field, the regularization prescription provides a correct result, which can be calculated from another consideration. Namely, for general N , the form factors of the field E_1 have a very simple property. It can be expressed in terms of form factors of the first energy field in the following way:

$$\langle E_1 | \{\beta\}, \{\beta'\} \rangle_{(n,n)} = \frac{\sum (e^{2\beta_j} - e^{2\beta'_j}) \sum (e^{-2\beta_j} - e^{-2\beta'_j})}{\sum (e^{\beta_j} + e^{\beta'_j}) \sum (e^{-\beta_j} + e^{-\beta'_j})} \langle \varepsilon_1 | \{\beta\}, \{\beta'\} \rangle_{(n,n)}. \tag{44}$$

This equation comes from the fact that in CFT [27], the energy field ε_1 has a null vector at level 2. This means that there is a linear relation between the descendants [19] $W_{-2}^{(3)} \varepsilon_1$ and $L_{-1} W_{-1}^{(3)} \varepsilon_1$. Now we have to take into account the fact that the modes $W_{-2}^{(3)}$ of the $W_3(z)$ current act on the energy field ε_1 as the spin-2 integral of motion. Due to its even spin, it is odd with respect to the charge conjugation transformation and have the form

$$\langle W_{-2}^{(3)} \varepsilon_1 | \{\beta\}, \{\beta'\} \rangle_{(n,n)} = \sum^n (e^{2\beta_j} - e^{2\beta'_j}) \langle \varepsilon_1 | \{\beta\}, \{\beta'\} \rangle_{(n,n)}. \tag{45}$$

This completes the explanation of equation (44).

⁶ The same is true for the Z_N charged fields $\psi_k \bar{\psi}_k$ and $\Phi_{2k,2k}^{(2)}$.

⁷ This theory can also be considered as a well-known sine-Gordon model in the reflectionless point, with the parameter $\beta^2 = \frac{1}{3}$. The form factors of the exponential fields in this case can be extracted from the results of [22, 33].

We propose to study another relation, following from the null vector conditions, in a separate publication.

4.3. Explicit expressions for form factors

Finally, let us derive explicit expressions for the form factors, which will be used in the correlation function studies. With definitions (27)–(31), equation (27) for the disorder fields has a conventional form:

$$\begin{aligned} \langle \Phi_{m,-m}^{(k)} | \{\beta\}, \{\beta'\} \rangle_{(n,n)} &= \frac{(-1)^n}{(2 \sin \frac{\pi}{N})^n} e^{\frac{m}{N} \sum (\beta_j - \beta'_j)} \sum_{\{a_j, b_j\}} \prod_j a_j b_j e^{\frac{i\pi}{2N} ((k+1)(a_j - b_j) - m(a_j + b_j))} \\ &\times \langle \mathcal{Z}_{a_1}(\beta_1) \cdots \mathcal{Z}_{b_n}^\dagger(\beta'_n) \rangle \langle \Phi_{m,-m}^{(k)} \rangle. \end{aligned} \quad (46)$$

As an example, we write down the explicit expressions for the first form factors of the disorder operator:

$$\begin{aligned} \langle \mu_k | 0 \rangle &= \langle \mu_k \rangle, \\ \langle \mu_k | \beta_1, \beta'_1 \rangle_{1 \uparrow 1} &= -\zeta^\dagger(\beta_1 - \beta'_1) \langle \mu_k \rangle \{k\} e^{\frac{N-2k}{2N}(\beta_1 - \beta'_1)}, \\ \langle \mu_k | \beta_1, \beta_2, \beta'_1, \beta'_2 \rangle_{1 \uparrow 1 \uparrow 1 \uparrow 1} &= \zeta(\beta_{12}) \zeta(\beta'_{12}) \prod_{ij}^2 \zeta^\dagger(\beta_i - \beta'_j) \\ &\times \frac{1}{4} \langle \mu_k \rangle \{k\}^2 e^{\frac{k-N}{N}(\beta_1 + \beta_2) - \frac{k}{N}(\beta'_1 + \beta'_2)} \\ &\times \left(\nu_1 \tau_1 + \frac{\{2\}\{k+1\}}{\{1\}\{k\}} \nu_2 + \frac{\{2\}\{k-1\}}{\{1\}\{k\}} \tau_2 \right), \end{aligned} \quad (47)$$

where ν_j and τ_j are j th symmetric polynomials of the variables e^{β_i} and $e^{\beta'_i}$, respectively, and we introduced the notation $\{a\} = \sin(\frac{\pi a}{N})$. In equation (47), we have written explicitly the normalization of scaling fields to its VEV [33]. The exact values of VEVs for the fields important for us [7] will be written below. It is rather a direct task to find out expressions for the higher particle form factors. However, in our numerical computations, we will not need it.

Another simplest example is the case of parafermionic currents. Our candidate for the parafermionic current form factors can be easily found by using equation (28):

$$\begin{aligned} \langle \psi | \beta'_1 \rangle_1 &= C_\psi e^{\frac{N-1}{N} \beta'_1}, \\ \langle \psi | \beta_1, \beta'_1, \beta'_2 \rangle_{1 \uparrow 1 \uparrow 1} &= -C_\psi \{2\} \zeta(\beta'_{12}) \prod_j^2 \zeta^\dagger(\beta_1 - \beta'_j) e^{\frac{1}{N} \beta_1 + \frac{N-2}{N}(\beta'_1 + \beta'_2)}, \\ \langle \psi | \beta_1, \beta_2, \beta'_1, \beta'_2, \beta'_3 \rangle_{1 \uparrow 1 \uparrow 1 \uparrow 1 \uparrow 1} &= \frac{1}{4} C_\psi \{2\}^2 \zeta(\beta_{12}) \prod_{i < j}^3 \zeta(\beta'_{ij}) \\ &\times \prod_i^2 \prod_j^3 \zeta^\dagger(\beta_i - \beta'_j) e^{(\frac{1}{N} - \frac{1}{2})(\beta_1 + \beta_2) - \frac{1}{N}(\beta'_1 + \beta'_2 + \beta'_3)} \\ &\times \left(\tau_2 + \nu_1 \tau_1 + \nu_2 \left(1 + 2 \cos \frac{2\pi}{N} \right) \right), \end{aligned} \quad (48)$$

etc. By C_ψ we denoted the normalization factor, determining the one-particle form factor. We will fix it explicitly in equation (71).

5. Correlation functions

In this section, we develop the conformal perturbation theory and compare long- and short-distance asymptotics of correlation functions of scaling fields. Our aim is to demonstrate that the asymptotics are in agreement with each other at the intermediate distances and, therefore, give an effective description of correlation functions at all scales. See [1, 7, 31, 40, 41] for other results in this direction.

Before proceeding further, let us make an important comment. The long-distance behavior of correlation functions is described in terms of the mass of the lightest particle parameter. The exact relation between the mass and the coupling constant can be derived via the TBA technique [2]. For the sine-Gordon model, this was done in Zamolodchikov's paper [3]. In our case, the mass–coupling constant relation can be found from the results of [35]. To simplify expressions, here and below we use the notations

$$\gamma(a) = \frac{\Gamma(a)}{\Gamma(1-a)}, \quad u = \frac{1}{N+2}, \quad \kappa = M \frac{\Gamma(2/N)\Gamma(1-1/N)}{\Gamma(1/N)}. \quad (49)$$

Applying these definitions, the explicit relation between the parameters M and λ is given as follows:

$$(2\pi\lambda)^2 = \kappa^{4(1-2u)}\gamma(u)\gamma(3u). \quad (50)$$

5.1. Conformal perturbation theory and exact VEVs

In this subsection, we consider the basic notions of the conformal perturbation theory [1] and give explicit values of the VEVs [7, 8] of the fields, which we will need in the computations.

In what follows, we develop the conformal perturbation theory [1] for the two-point functions of scaling fields Φ_a and Φ_b :

$$\langle \Phi_a(z, \bar{z})\Phi_b(0) \rangle = \sum_l C_{a,b}^{O_l}(z, \bar{z})\langle O_l(0) \rangle, \quad (51)$$

where $C_{a,b}^{O_l}(z, \bar{z})$ are the structure functions and operators O_l form the basis in the space of scaling fields. Functions $C_{a,b}^{O_l}(z, \bar{z})$ can be expanded in the perturbation series

$$C_{a,b}^{O_l}(z, \bar{z}) = |z|^{2\Delta_{O_l}} z^{-\Delta_{\Phi_a} - \Delta_{\Phi_b}} \bar{z}^{-\bar{\Delta}_{\Phi_a} - \bar{\Delta}_{\Phi_b}} \left(C_{a,b}^{(0)O_l} + \lambda|z|^{2(1-D_1)} C_{a,b}^{(1)O_l} + \dots \right),$$

where the coefficients $C_{a,b}^{(0)O_l}$ are the structure constants from the conformal fields theory, while the first-order corrections $C_{a,b}^{(1)O_l}$ can be expressed through the integrals of correlation functions in CFT:

$$C_{a,b}^{(1)O_l} = \int d^2y \langle \Phi_a(0)\Phi_b(1)\varepsilon_1(y, \bar{y})O_l(\infty) \rangle. \quad (52)$$

The vacuum expectation values $\langle O_l \rangle$ of the scaling fields O_l , appearing in equation (51), have a non-perturbative nature [1]. These fundamental quantities depend on the normalization prescription for the fields. We find VEVs, assuming that the scaling fields satisfy the standard conformal normalization prescription:

$$\langle O(z, \bar{z})O^\dagger(0) \rangle = |z|^{-4\Delta_o} + \dots, \quad |z| \rightarrow 0. \quad (53)$$

The exact VEVs for physically important operators $\Phi_{m,\bar{m}}^{(k)}$ and $E_k = W_{-1}^{(3)}\bar{W}_{-1}^{(3)}\varepsilon_k$ in the scaling Z_N Ising model were found in [7, 8], following the approach of [4, 26, 35]. The vacuum expectation values for the disorder fields are

$$\langle \mu_k \rangle = \kappa^{2d_k} \exp \int_0^\infty \frac{dt}{t} \left(\frac{\sinh kut \sinh(N-k)ut}{\sinh t \tanh Nut} - 2d_k e^{-2t} \right). \quad (54)$$

It is interesting that for even values of k , the above integral can be calculated, giving

$$\langle \mu_{2k} \rangle = \kappa^{2d_{2k}} \left(\frac{\gamma\left(\frac{1}{N+2}\right)}{\gamma\left(\frac{2k+1}{N+2}\right)} \right)^{1/2} \left(\frac{N+2}{N} \right)^{\frac{k(N-2k)}{N}} \prod_{i=1}^k \frac{\gamma\left(\frac{2i-1}{N}\right)}{\gamma\left(\frac{2i-1}{N+2}\right)}. \quad (55)$$

For a general case, $\Phi_{m,-m}^{(k)} = (\psi^\dagger)^l (\bar{\psi})^l \mu_k$, where $m = k - 2l$, the vacuum expectation values of fields are elegantly expressed in terms of $\langle \mu_k \rangle$ as

$$\frac{\langle \Phi_{m,-m}^{(k)} \rangle}{\langle \mu_k \rangle} = \left(\frac{\kappa(N+2)}{N} \right)^{\frac{k^2-m^2}{2N}} \prod_{i=0}^{l-1} \frac{(i+1)\gamma\left(\frac{i+1}{N}\right)}{(k-i)\gamma\left(\frac{k-i}{N}\right)}. \quad (56)$$

In particular, the VEVs of thermal fields $\varepsilon_k = (\psi)^k (\bar{\psi})^k \mu_{2k}$ correspond to the $m = 0$ case in the above equation:

$$\langle \varepsilon_k \rangle = \kappa^{2D_k} \left(\frac{N+2}{N} \right)^k \frac{(k!)^2}{(2k)!} \left(\frac{\gamma((2k+1)u)}{\gamma(u)} \right)^{\frac{1}{2}} \times \prod_{i=1}^k \frac{\gamma^2\left(\frac{i}{N}\right)}{\gamma\left(\frac{2i}{N}\right)\gamma((2i+1)u)}. \quad (57)$$

We also obtained the exact result for the expectation values of the normalized descendent fields $E_k = W_{-1}^{(3)} \bar{W}_{-1}^{(3)} \varepsilon_k$:

$$\frac{\langle E_k \rangle}{\langle \varepsilon_k \rangle} = \kappa^2 \frac{(N+2)^2}{2N} \frac{\Gamma^2\left(1 + \frac{k+1}{N}\right)\Gamma^2\left(1 - \frac{k}{N}\right)}{\Gamma^2\left(1 - \frac{k+1}{N}\right)\Gamma^2\left(1 + \frac{k}{N}\right)} \frac{\Gamma\left(1 + \frac{2k}{N}\right)\Gamma\left(1 - \frac{2k+2}{N}\right)}{\Gamma\left(2 - \frac{2k}{N}\right)\Gamma\left(2 + \frac{2k+2}{N}\right)}. \quad (58)$$

5.2. Correlation functions $\langle \mu_1(x)\mu_1(0) \rangle$

In our previous paper [7], we studied long- and short-distance asymptotics of the correlation functions $\langle \sigma_1(x)\sigma_1^\dagger(0) \rangle$ and $\langle \mu_1(x)\mu_1^\dagger(0) \rangle$ of order and disorder fields. For completeness, we collect the correspondent ultraviolet expansion data in appendix A. We found that IR and UV asymptotics of the correlators match at the intermediate distances. This confirms our expressions for VEVs as well as form factor expressions.

In this subsection, we want to discuss a more complicate case of the correlation function, including two disorder fields $\langle \mu_1(x)\mu_1(0) \rangle$. We found that studying this case is rather instructive, since it gives an example, where the resonance fields (3) and (4) appear at the short-distance expansion for all integer $N = 2, 3, \dots$

The long-distance expansion of this correlation function is provided in a standard way. In a two-particle form factor approximation, we have the spectral decomposition

$$\begin{aligned} \langle \mu_1(z, \bar{z})\mu_1(0) \rangle &= \langle \mu_1 \rangle^2 + \int d\theta_1 d\theta_2 (\langle \mu(z, \bar{z})|\theta_1, \theta_2 \rangle_{1,1^\dagger} \langle \mu^\dagger(0)|\theta_2 + i\pi, \theta_1 + i\pi \rangle_{1,1^\dagger} \\ &+ \langle \mu(z, \bar{z})|\theta_1, \theta_2 \rangle_{1^\dagger,1} \langle \mu^\dagger(0)|\theta_2 + i\pi, \theta_1 + i\pi \rangle_{1^\dagger,1} + \dots \end{aligned} \quad (59)$$

Studying the short-distance behavior is more involved. Indeed, we have the following leading terms in the conformal perturbation theory expansion (51):

$$\begin{aligned} \langle \mu_1(z, \bar{z})\mu_1(0) \rangle &= C_{\mu_1\mu_1}^{\mu_2}(r) \langle \mu_2 \rangle + C_{\mu_1\mu_1}^\eta(r) \langle \eta \rangle \\ &+ C_{\mu_1\mu_1}^{\psi\bar{\psi}}(r) \langle \psi\bar{\psi}^\dagger \rangle + \dots, \quad |z| = r, \end{aligned} \quad (60)$$

where we introduced the notation η for the field $\eta = \Phi_{4,2}^{(2)}$. The leading contribution to the correlator (60) at the short distances comes from the zero-order term $C_{\mu_1\mu_1}^{(0)\mu_2}(r) \langle \mu_2 \rangle$ of the perturbation theory. It can be computed using CFT structure constants, found in [9], and the exact vacuum expectation value for the second disorder field μ_2 (54):

$$C_{\mu_1\mu_1}^{(0)\mu_2}(r) \langle \mu_2 \rangle = r^{-2d_1} \frac{\gamma\left(\frac{1}{N}\right)}{\gamma(2u)} (r\kappa)^2 \frac{N-2}{N(N+2)} \left(\frac{N+2}{N} \right)^{\frac{N-2}{N}}, \quad u = \frac{1}{N+2}. \quad (61)$$

At the first-order perturbation theory, we have the term including the field η . The contribution from this field to the perturbative expansion can be expressed in terms of the product of the CFT structure constants $C_{\mu_1\mu_1}^{(0)\mu_2} C_{\mu_2\varepsilon_1}^{(0)\eta}$, multiplied by the simple two-dimensional integral over y from the correlation function

$$\langle \mu_1(0)\mu_1(1)\varepsilon_1(y, \bar{y})\eta(\infty) \rangle_{\text{CFT}} = |y|^{2u}|1-y|^{2u}.$$

Using the exact result (56) for the VEV of the field $\eta = \Phi_{2,-2}^{(4)}$, we obtain the analytic result for this term in the first-order approximation:

$$C_{\mu_1\mu_1}^\eta(r)\langle \eta \rangle = \frac{r^{-2d_1}}{16} (r\kappa)^2 \frac{\gamma\left(\frac{1}{N}\right)\gamma\left(\frac{3}{N}\right)\gamma^3(u)u^2}{(1+2u)^2\gamma(3u)\gamma^2(2u)} \left(\frac{1}{Nu}\right)^{2\frac{N-1}{N}}. \quad (62)$$

A more difficult task is to compute the first-order correction to the structure function $C_{\mu_1\mu_1}^{\mu_2}(r)$ and the contribution, coming from the term $\psi\bar{\psi}^\dagger$. We note that the conformal dimensions of the fields $\psi\bar{\psi}^\dagger$ and μ_2 satisfy the relation

$$2\Delta_\psi - 2\Delta_{\mu_2} = 2\frac{N}{N+2} = 2(1 - \Delta_{\varepsilon_1}). \quad (63)$$

The first-order resonance condition (3) now appears at the short-distance expansion. Equation (63) basically means that two terms in the perturbation theory will have the same powers of r . Whenever this happens, we expect that divergences coming from the contributions from the term $C_{\mu_1\mu_1}^{\psi\bar{\psi}^\dagger}(r)\langle \psi\bar{\psi}^\dagger \rangle$ should be canceled by divergences appearing from the first-order correction to the structure function $C_{\mu_1\mu_1}^{(1)\mu_2}(r)\langle \mu_2 \rangle$. This phenomenon, in general, should lead to logarithmic terms in a perturbative expansion.

The appropriate contributions to the resonance terms can be computed within the Z_N Ising model settings by providing the analytic continuation in the parameter N , as was discussed before, i.e. we can compute the contributions after the change $N \rightarrow N + \epsilon$ and then consider the limit $\epsilon \rightarrow 0$. To check the validity of this approach, we also perform the computations in a different method. We use the fact that parafermionic CFT with the central charge (1) belongs to the series of W algebra symmetric unitary minimal CFT $\mathcal{WM}_N^{(p)}$, where the parameter p is chosen to be $p = N + 1$ [27]. The field theory (2), from the view point of this model, is the simplest representative of the series of integrable perturbations of $\mathcal{WM}_N^{(p)}$ minimal models by the primary field with the conformal dimension $D_1^{(p)} = 1 - \frac{N}{p+1}$, which, in our case, coincides with the first energy operator. To study the contributions of the resonance fields, we can do the deformation of $\mathcal{WM}_N^{(p)}$ by changing the parameter p in such a way that the variable $u = 1/(p + 1)$ would have the form

$$u_\epsilon = \frac{1}{N+2} - \epsilon.$$

For small non-zero ϵ , the expressions for the contributions of the resonance fields are finite and well defined. The result of computations can be written in the following way:

$$\begin{aligned} C_{\mu_1\mu_1}^{(1)\mu_2}(r)\langle \mu_2 \rangle &= r^{-2d_1} (r\kappa)^{\frac{2}{N}(N-2+(N+2)u_\epsilon)} \frac{2}{N^2} \frac{(1 - (N+1)u_\epsilon)^2}{(1 - Nu_\epsilon)^2} \\ &\times \frac{\gamma\left(\frac{1}{N}\right)\gamma(Nu_\epsilon)\gamma(u_\epsilon)\gamma^2\left(\frac{1+Nu_\epsilon}{2}\right)\gamma^2\left(\frac{1-(N-2)u_\epsilon}{2}\right)\gamma\left(\frac{1-(N+2)u_\epsilon}{2}\right)}{\gamma(2u_\epsilon)\gamma((N+1)u_\epsilon)(Nu_\epsilon)^{\frac{N-2}{N}}} \\ &\times (1 + \epsilon(N+2)\chi_0 + \mathcal{O}(\epsilon^2)). \end{aligned} \quad (64)$$

To simplify the resulting expression, we introduced a shorthand notation for the term in the last line:

$$\chi_0 = -\frac{1}{2} \left(\psi \left(\frac{1}{2} + \frac{(N-2)u}{2} \right) + \bar{\psi} \left(\frac{1}{2} - \frac{(N-2)u}{2} \right) - \frac{2}{N} \gamma_E + (N+2) \log \left(\frac{N+2}{N} \right) \right).$$

The symbol γ_E is reserved for Euler's constant and ψ stands for the logarithmic derivative of the gamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$. (Unfortunately, this standard notation conflicts with the usual symbol, which we choose for the parafermionic current $\psi(z)$.)

The contribution, which comes from the term proportional to $\langle \psi \bar{\psi}^\dagger \rangle$, is found to be of a similar form:

$$\begin{aligned}
 C_{\mu_1 \mu_1}^{\psi \bar{\psi}^\dagger}(r) \langle \psi \bar{\psi}^\dagger \rangle &= r^{-2d_1} (r\kappa)^{2\frac{N-2}{N}(2-(N+2)u_\epsilon)} \frac{(1-(N+1)u_\epsilon)^2}{(1-2u_\epsilon)^2} \\
 &\times \left(\frac{1}{Nu_\epsilon} \right)^{2\frac{N-1}{N}} \frac{\gamma(Nu_\epsilon)}{\gamma(u_\epsilon)} \gamma\left(\frac{1-Nu_\epsilon}{2}\right) \gamma\left(\frac{1-(N-2)u_\epsilon}{2}\right) \\
 &\times \gamma\left(\frac{1}{Nu_\epsilon} - 1 - \frac{1}{N}\right) \gamma\left(\frac{N+2}{N} - \frac{1}{Nu_\epsilon}\right) \\
 &\times (1 + \epsilon(N+2)\chi_1 + O(\epsilon^2)), \tag{65}
 \end{aligned}$$

where

$$\chi_1 = -\frac{1}{2} \left(\psi\left(\frac{1}{2} + \frac{Nu}{2}\right) - \psi\left(\frac{1}{2} - \frac{Nu}{2}\right) \right) + \chi_0.$$

Now we look for the regularized expressions in the limit $\epsilon \rightarrow 0$.

It can be derived from equations (64) and (65) that both terms have a single pole at $\epsilon = 0$. However, as we expected from a general consideration, the sum of residues at this pole is zero, and the total contribution of these resonance terms to the correlation functions is well defined in the $\epsilon \rightarrow 0$ limit. Computing the limit, we find that the correctly defined sum of two resonance terms is given as

$$\begin{aligned}
 [C_{\mu_1 \mu_1}^{(1)\mu_2}(r) \langle \mu_2 \rangle + C_{\mu_1 \mu_1}^{\psi \bar{\psi}^\dagger}(r) \langle \psi \bar{\psi}^\dagger \rangle]^{\text{reg}} &= \frac{\gamma\left(\frac{1}{N}\right)}{2N^2} r^{-2d_1} (r\kappa)^{2\frac{N-1}{N}} \left(\frac{N+2}{N}\right)^{\frac{N-2}{N}} \\
 &\times \left(8(\gamma_E - 1) + 4N \log r\kappa e^{\frac{3}{2}\gamma_E - 1} + 2(N+2) \left(\psi\left(\frac{1}{N}\right) + \psi\left(-\frac{1}{N}\right) \right) \right. \\
 &\left. - N \left(\psi\left(\frac{2}{N+2}\right) + \psi\left(\frac{N}{N+2}\right) \right) \right). \tag{66}
 \end{aligned}$$

The numerical data for long- and short-distance asymptotic expansions are shown in figure 5 for the case $N = 7$. The dashed curve is for UV asymptotics, while the full line denotes the form factor decomposition, up to two particles. We observe that there is the asymptotics match at the intermediate distances. We found that in the region $0.01 \leq Mr \leq 1$, the long- and short-distance asymptotics agree with a relative error of around 1%. This, rather good, numerical preciseness confirms our hypothesis on the identification of form factors, as well as the short-distance regularization prescription.

For other values of the parameter N , we provided similar numerical computations. We found that the relative error decreases with an increase of number N . For large N , the error becomes smaller. For example, for the $N = 11$ case, it is already less than 0.1%. The agreement between data can be further improved by taking into account higher particle form factors; however, this is beyond the scope of this paper.

In the limit of large parameters N , we find that our short-distance expansion function behaves as

$$\frac{\langle \mu_1(z, \bar{z}) \mu_1(0) \rangle}{\langle \mu_1 \rangle^2} = 1 + \frac{1}{N^2} \left(-2\Omega + \left(\Omega^2 - 4\Omega + \frac{9}{2} \right) M^2 r^2 \right) + \dots, \tag{67}$$

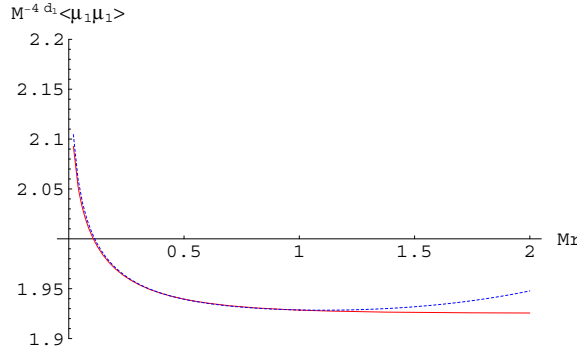


Figure 5. Correlation function of disorder fields at $N = 7$.

where $\Omega = \log\left(\frac{Mr e^{\gamma_E+1}}{2}\right)$. We checked, using the Mellin transform, that our form factor expression leads to the same expansion at the small distances up to the order N^{-3} . In the next orders, new corrections can appear from the higher particle form factors.

Another non-trivial test for the correctness of our expressions is the limit to the Ising model point $N = 2$. In the case, when $N \rightarrow 2$, our ultraviolet asymptotics leads to the following answer:

$$\langle \mu_1(z, \bar{z}) \mu_1(0) \rangle = \frac{1}{r^{\frac{1}{4}}} \left(1 - \frac{Mr}{2} \log \frac{Mr e^{\gamma_E}}{8} + \dots \right), \tag{68}$$

which agrees with the known short-distance expansion for the Ising model disorder parameter correlation function [15, 36].

5.3. Correlation functions of parafermionic currents

In this subsection, we would like to compare short- and long-distance asymptotics for the two-point correlation functions of parafermionic currents ψ and ψ^\dagger at the region $Mr \sim 1$. We think that a consideration of this case might be interesting because the correlator of free fermions is first of all one of the most simplest in the Ising model. Unlike the case of order and disorder fields, it is given exactly by the Bessel function, i.e. by the solution of a linear equation, which is simpler than the Painleve equation [15, 36]. On the other hand, this is one of the few examples of correlation functions of operators with fractional spins.

We recall that in the unperturbed CFT, the currents $\psi(z)$ and (ψ^\dagger) have conformal dimensions $(\Delta_1, 0)$ and $(\Delta_{N-1}, 0)$, defined by equation (12). In the ultraviolet region the correlation function of these fields is expected to have a form (see equation (18))

$$\langle \psi(z, \bar{z}) \psi^\dagger(0) \rangle = z^{-2\Delta_1} + \dots, \quad |z| \rightarrow 0, \tag{69}$$

which is a natural definition of the conformal normalization in the given case of operators with non-trivial spins.

5.3.1. General N case. The leading contribution to the infrared asymptotics of the correlation function of parafermionic currents comes from the one-particle form factor approximation. We will see that it already leads to a nice agreement between the asymptotics at the intermediate distances. Using the integral representation for the modified Bessel function, we can express

the one-particle contribution in the following analytic form:

$$\langle \psi(z, \bar{z}) \psi^\dagger(0) \rangle = 2C_\psi^2 K_{\frac{N-1}{N}}(\text{Mr}) + \dots, \quad z = r. \quad (70)$$

For simplicity, we choose here and below the space coordinate to be zero, which corresponds to the real $z = r$.

In equation (70), we took into account the fact that for the fields with spin, the vacuum expectation values vanish. Our proposal for the exact value of the multiple C_ψ^2 , leading to the conformal normalization (69) of parafermionic currents, is

$$C_\psi^2 = \frac{\Gamma(1 + \frac{1}{N})}{\Gamma(1 - \frac{1}{N})} \left(\frac{N+2}{N} \right)^{2\frac{N-1}{N}} \kappa^{2\frac{N-1}{N}} S_2^2 \left(2\pi + \frac{2\pi}{N} \right). \quad (71)$$

The constant $S_2(2\pi + \frac{2\pi}{N})$ is defined by equation (B.1) in appendix B. This expression can be obtained, following the ideas of [34]. It comes naturally from the analysis of the divergence in the VEV of the operator $\psi \bar{\psi}^\dagger$, which can be effectively provided in the framework of the W -symmetric CFT, perturbed by the adjoint field, by using the results of paper [35]. In another way, it can be obtained by analyzing the deformed parafermionic currents' normalization; see [8]. Further, we will see that this normalization coincides with the known exact results for $N = 2$ and $N \rightarrow \infty$ cases. We also establish numerical checks, by matching the long- and short-distance asymptotics for correlators for arbitrary N .

From the general arguments of the conformal perturbation theory [1], we found that the short-distance expansion of the correlator of parafermionic currents has the form

$$\begin{aligned} \langle \psi(z, \bar{z}) \psi^\dagger(0) \rangle = & \frac{1}{z^{2\Delta_1}} \left(1 + (\kappa r)^2 A_1 + (\kappa r)^{\frac{4N}{N+2}} A_2 + (\kappa r)^{\frac{4(N+3)}{N+2}} A_3 + (\kappa r)^{\frac{4N}{N+2}+2} A_4 \right. \\ & \left. + (\kappa r)^{\frac{4(N+1)}{N+2}+2} A_5 + (\kappa r)^{\frac{6(N+4)}{N+2}} A_6 + \dots \right). \end{aligned} \quad (72)$$

Here and below in this subsection, we use in the short-distance expansions the notation $r = |z|$. For the computation of the coefficients A_j ($j = 1, \dots, 6$) in this equation, it is convenient to use the quantum equations of motion (37):

$$\frac{\partial}{\partial \bar{z}} \psi(z, \bar{z}) = \lambda \sqrt{\frac{2}{N}} \Phi_{2,0}^{(2)}(z, \bar{z}). \quad (73)$$

We recall that the fields $\Phi_{2,0}^{(2)} = (\psi)\epsilon_1$ and $\Phi_{-2,0}^{(2)} = (\psi^\dagger)\epsilon_1$ have different left and right conformal dimensions $(\Delta_{\mu_2}, \Delta_{\epsilon_1})$ defined by equations (7) and (23) respectively.

To define coefficient A_1 , we integrate over \bar{z} the first term in the decomposition of the following two-point correlation function:

$$\langle \bar{\partial} \psi(z, \bar{z}) \psi^\dagger(\zeta, \bar{\zeta}) \rangle = \lambda \sqrt{\frac{2}{N}} \langle \Phi_{2,0}^{(2)}(z, \bar{z}) \psi^\dagger(\zeta, \bar{\zeta}) \rangle. \quad (74)$$

In this prescription, the coefficient A_1 is computed in a simple way from the CFT three-point correlation function:

$$\langle \Phi_{2,0}^{(2)}(0) \psi^\dagger(1) \epsilon_1(\infty) \rangle_{\text{CFT}},$$

multiplied by the expectation value of the field ϵ_1 . An effective method of computation of other coefficients in equation (72) is to integrate twice the series expansion for the following correlation function:

$$\langle \bar{\partial} \psi(z, \bar{z}) \bar{\partial} \psi^\dagger(\zeta, \bar{\zeta}) \rangle = \frac{2\lambda^2}{N} \langle \Phi_{2,0}^{(2)}(z, \bar{z}) \Phi_{-2,0}^{(2)}(\zeta, \bar{\zeta}) \rangle. \quad (75)$$

More explicitly, starting from the conformal perturbation theory expansion for the two-point correlator

$$\langle \Phi_{2,0}^{(2)}(z, \bar{z}) \Phi_{-2,0}^{(2)}(\zeta, \bar{\zeta}) \rangle = \frac{|z - \zeta|^{\frac{4N}{N+2}}}{(z - \zeta)^{2\Delta_1} (\bar{z} - \bar{\zeta})^2} \times (C^I(r) + C^{\epsilon_1}(r)\langle\epsilon_1\rangle + C^{\epsilon_2}(r)\langle\epsilon_2\rangle + C^{E_1}(r)\langle E_1\rangle + C^{\epsilon_3}(r)\langle\epsilon_3\rangle + \dots), \quad (76)$$

we obtain, by integration over \bar{z} and $\bar{\zeta}$, the coefficient A_2 from the coefficient C^I . In a similar manner, the coefficients A_3, A_5 and A_6 are related, correspondingly, to the zeroth-order structure functions $C^{\epsilon_2}\langle\epsilon_2\rangle, C^{E_1}\langle E_1\rangle$ and $C^{\epsilon_3}\langle\epsilon_3\rangle$ respectively. The coefficient A_4 is related to the first-order correction term $C^{(1)\epsilon_1}\langle\epsilon_1\rangle$.

Using the free field realization for conformal primary fields [9, 11], one can find an integral representation for the leading corrections to the structure functions in expansion (75). The integrals in the first-order perturbation theory can be taken by applying the technique from [37]. The exact vacuum expectation values for the energy fields $\langle\epsilon_k\rangle (k = 1, 2, 3)$ and E_1 are given in equations (57) and (58) respectively. The results for numerical coefficients A_k can be written as

$$\begin{aligned} A_1 &= -\frac{N+2}{2N^2} \frac{\gamma\left(\frac{1}{N}\right)^2}{\gamma\left(\frac{2}{N}\right)}, \\ A_2 &= \frac{(N+2)^2}{4N^2(N-2)} \gamma(u)\gamma(3u), \\ A_3 &= -\frac{1}{72} \frac{(N+2)^4}{(N+3)(N+4)N^3} \frac{\gamma(u)}{\gamma(3u)} \frac{\gamma^2\left(\frac{1}{N}\right)\gamma\left(\frac{2}{N}\right)}{\gamma\left(\frac{4}{N}\right)}, \\ A_4 &= -\frac{(N+2)^3}{16N^3(3N+2)} \frac{\gamma^2\left(\frac{1}{N}\right)}{\gamma\left(\frac{2}{N}\right)} \gamma(u)\gamma(3u) J_N, \\ A_5 &= -\frac{(N+2)^5(N-4)}{8(N+1)(3N+4)} \frac{1}{(N+4)^2(N-2)^2N} \frac{\gamma^2\left(\frac{2}{N}\right)}{\gamma\left(\frac{4}{N}\right)} \frac{\gamma(4u)\gamma^2(u)}{\gamma^2(2u)}, \\ A_6 &= -\frac{1}{2400} \frac{(N+2)^5}{N^4(N+4)(N+5)(N+6)^2} \frac{\gamma^2\left(\frac{1}{N}\right)\gamma^2\left(\frac{2}{N}\right)\gamma^2\left(\frac{3}{N}\right)}{\gamma\left(\frac{2}{N}\right)\gamma\left(\frac{4}{N}\right)\gamma\left(\frac{6}{N}\right)} \frac{\gamma^2(2u)\gamma^2(u)}{\gamma(3u)\gamma(4u)\gamma(5u)}. \end{aligned} \quad (77)$$

Note that the coefficient A_4 was computed analytically only for particular values of N . This is because the multiple J_N in this coefficient represents the contribution, which is given by the rather complicate integral over the plane, from the CFT correlation function

$$J_N = \frac{1}{\pi} \int d^2y \langle \Phi_{2,0}^{(2)}(1) \Phi_{-2,0}^{(2)}(0) \epsilon_1(y, \bar{y}) \epsilon_1(\infty) \rangle_{\text{CFT}}. \quad (78)$$

The asymptotics of the integral J_N , considered as a function of N , will be given for large values of N further in equation (84).

In the following subsection, we discuss the particular cases of N , for which the correlation function has specific features, including resonances.

5.3.2. Ising model $N = 2$ case. We first consider the consistency of our formulae for $N = 2$ with the known results from the Ising model. According to the fusion rules in the Ising model case, there are no higher energy fields in the short-distance expansion. So, we have to restrict ourselves to the first two terms coming with the coefficients A_1 and A_2 . A formal substitution

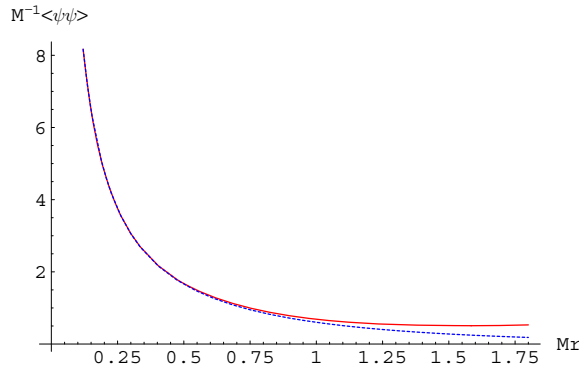


Figure 6. Parafermionic correlators at $N = 2.00001$.

$N = 2$ leads to a divergence which is, of course, expected from our previous consideration of the resonance fields. When we consider the limit $N \rightarrow 2$, we find that the residues at the pole $\frac{1}{N-2}$ are canceled, and the resulting expansion for the correlator has a logarithmic scaling. Let us assume that $z = r$ is chosen to be real. Then, the $N \rightarrow 2$ limiting value for the correlator will have the form

$$\begin{aligned} \langle \psi(z, \bar{z}) \psi(0) \rangle &= \lim_{N \rightarrow 2} r^{-2 \frac{N-1}{N}} \left(1 + (\kappa r)^2 A_1 + (\kappa r)^{\frac{4N}{N+2}} A_2 + \dots \right) \\ &= \frac{1}{r} \left(1 + \frac{(Mr)^2}{2} \log \left(\frac{Mr}{2} e^{\gamma_E - \frac{1}{2}} \right) + \dots \right). \end{aligned} \tag{79}$$

This equation agrees with the form factor ultraviolet expansion, where the exact correlator is given in terms of the modified Bessel function $K_1(r)$. In the numerical computations, we can formally put N in the general coefficients A_1 and A_2 to be close to 2. Then, the long- and short-distance expressions would match at the intermediate distances, as is expected. Since the infrared expansion result is an exact one, it is instructive to see for our short-distance formulae and efficiency of the analytic continuation prescription. For example, the curves depicted in figure 6 correspond to $N = 2.00001$.

In the figure, the dashed lines are given by the first three terms in the UV decomposition. The full line, here and below, represents the one-particle form factor expansion. The dependence on N , in a vicinity of the Ising model case value of $N = 2$, is smooth. Similarity between long- and short-distance asymptotics of the Ising model case serves as one of the confirmation, supporting the normalization constant C_ψ^2 (71).

5.3.3. *Three-state Potts model $N = 3$ case.* The model for $N = 3$ coincides with the scaling three-state Potts model. Up to the normalization factors, the exact form factors for parafermionic currents (as well as for other primaries) were computed for this model in [29]. The correlation functions of order and disorder fields in this model were also studied in [7, 31].

According to the fusion rules for the $N = 3$ case, there are no energy fields ε_2 and ε_3 and we have to omit terms with the coefficients A_3 and A_6 . We show in figure 7 that even first three terms in the short-distance expansion lead to a rather good agreement between correlation function asymptotics.

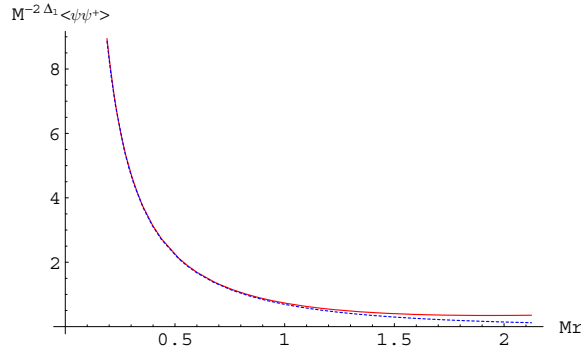


Figure 7. Parafermionic correlators at $N = 3$.

In this case, we can improve the exact computations further. In particular, the coefficient A_4 can be calculated exactly, since the integral J_3 can be taken analytically as

$$J_3 = -\frac{9\sqrt{5}\Gamma^5(\frac{3}{5})\Gamma^4(\frac{1}{5})}{16\pi^2\Gamma^3(\frac{2}{5})\Gamma^2(-\frac{1}{5})}. \tag{80}$$

Then, the contribution from this first-order perturbation theory term is given by the expression

$$(z\kappa)^{2+4\frac{N}{N+2}} A_4|_{N=3, z=r} = \frac{5}{3 \times 11 \times 2^6} \frac{\gamma^2(\frac{1}{3})\gamma^4(\frac{1}{5})}{\gamma(\frac{2}{3})\gamma^5(\frac{2}{5})} (r\kappa)^{\frac{22}{5}}. \tag{81}$$

It is also possible to compute the second-order perturbation correction to the identity operator in expansion (72), as well as the contribution of the descendant field $T\bar{T}$; these VEVs can be computed from the results of [38, 39]. With these corrections, and with the term taking the coefficient A_5 into account, the long- and short-distance expressions agree up to the distances $Mr = 3$.

5.3.4. $N = 4$ model. The model for $N = 4$ describes a particular case of the Ahskin–Teller model. For the parafermionic fields, proceeding as before, we find that the resonances appear at the short-distance expansion, as well.

Namely, providing an analytic continuation of general expressions with parameter N , we can easily see that the term with the coefficient A_3 diverges at $N \rightarrow 4$. However, the divergence, coming from this term, is canceled by the singular part of the term with the coefficient A_4 , since the integral J_N has the following expansion, when N approaches 4:

$$J_N = \frac{4}{3} \frac{1}{N-4} + \frac{5}{18} + O(N-4). \tag{82}$$

In this case, we meet the resonance condition (3) between the operators ϵ_2 and ϵ_1 . As in the Ising model, let us look for an analytic continuation in N of the general expression. Finding the regular, in the parameter N , expansion, we obtain

$$\lim_{N \rightarrow 4} ((\kappa r)^{4\frac{N+3}{N+2}} A_3 + (\kappa r)^{2+4\frac{N}{N+2}} A_4) = \frac{3}{2^6 \times 7} \frac{\gamma(\frac{1}{6})}{\gamma^2(\frac{3}{4})} (r\kappa)^{\frac{14}{3}} \log\left(\frac{r\kappa}{2} e^{\gamma^E - \frac{13}{28}}\right). \tag{83}$$

Again, the presence of fields, which are in the resonance, leads to the logarithmic dependence of the short-distance approximation. Let us note that there is no divergence at the term A_5 :

$$\lim_{N \rightarrow 4} (\kappa r)^{2+4Nu+4u} A_5 = -\frac{3^{\frac{1}{2}} \times 3^5}{2^{\frac{1}{3}} \times 2^{10} \times 5\pi} \Gamma^2\left(\frac{1}{3}\right) (\kappa r)^{\frac{16}{3}}.$$

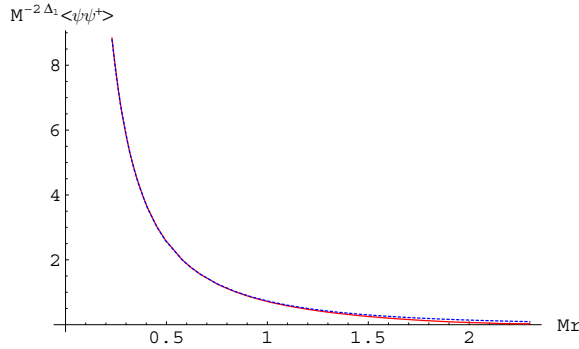


Figure 8. Parafermionic correlators at $N = 4$.

Substituting these regularized expressions into the short-distance expansion, we get a well-defined expansion. The numerical data (where the contribution from the second-order correction was taken into account) are shown in figure 8.

The relative error between the asymptotic values in the region $0.0001 < Mr < 1$ in our numerical computations is less than 1%.

5.3.5. *Large N case.* The nice feature of the Z_N models is that the correlation functions in the ultraviolet and infrared regions (72) are in agreement for an infinite set of models including those with an arbitrary integer $N \geq 2$. We propose in the large N limit the following approximation for the multiple J_N in equation (77):

$$J_N = \frac{18}{5N} \left(1 + \frac{32}{3N^2} + O\left(\frac{1}{N^3}\right) \right). \tag{84}$$

Taking the A_4 term into account, we have, for the asymptotic expansion for correlation functions at $N \rightarrow \infty$ in the vicinity of $r = 0$, the following expansion:

$$M^{-2\Delta_1} \langle \psi(z, \bar{z}) \psi^\dagger(0) \rangle|_{z=r} = \frac{1}{(Mr)^2} \left(T_0(Mr) + \frac{1}{N} T_1(Mr) + \dots \right). \tag{85}$$

Here, the functions $T_0(x)$ and $T_1(x)$ are given by the expressions, which agree with the one-particle form factor formulae for large N and small scales:

$$\begin{aligned} T_0(x) &= 1 - \frac{x^2}{4} - \frac{x^4}{16} \left(\log x + \gamma_E - \frac{3}{4} \right), \\ T_1(x) &= 2 \log x - \frac{x^2}{2} (1 + \log x) \\ &\quad + \frac{x^4}{16} \left(2 \left(\log \frac{e^\gamma}{2} - \frac{7}{4} \right) \log x + 2 \log^2 \frac{e^\gamma}{2} - 5 \log \frac{e^\gamma}{2} + \frac{13}{4} \right). \end{aligned} \tag{86}$$

With these conventions, both short- and long-distance expansions produce similar-looking curves for $N > 6$. The example of the correlation functions for the case $N = 11$ is given in figure 9.

To show the difference between the asymptotics, we also draw the rescaled correlators $r^{2\frac{N-1}{N}} \langle \psi \psi^\dagger \rangle$ in figure 10.

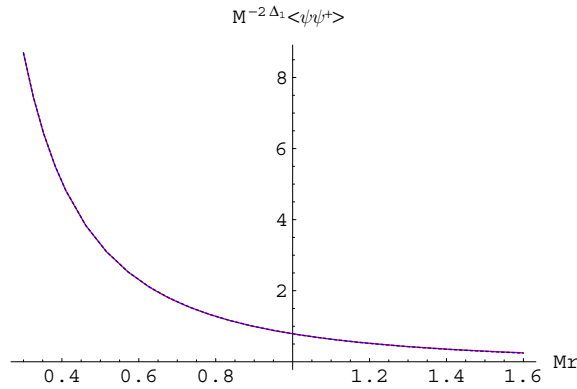


Figure 9. Parafermionic correlators at $N = 11$.

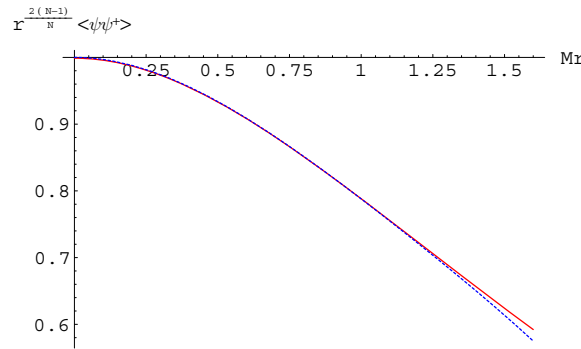


Figure 10. Rescaled parafermionic correlators at $N = 11$.

Let us note that the relative error for ultraviolet and infrared asymptotics at the region $0.000\,001 < Mr < 1$ becomes smaller with an increase of the number N . For example, for the $N = 7$ model, the error in this region is 0.8%, while for $N = 20$ it is already 0.08%. To deal with these small N cases, we considered the A_4 term more accurately. Namely, we provided the Pade approximation for the integral J_N (78) between the known and fixed points $N = 4$ and $N = \infty$, and found that the similarity between the asymptotics of the correlation functions for small N is within 1% at distances $0.000\,001 < Mr < 1$.

6. Concluding remarks

In this paper, we studied the correlation functions of disorder fields and parafermionic currents for scaling Z_N Ising models and found that the long- and short-distance asymptotics approach each other at the intermediate distances. On one hand, a similarity between ultraviolet and infrared asymptotics gives an effective way of studying the basic behavior of the correlation function of the theory at all distances. On the other hand, it confirms our construction of the form factors of the scaling fields.

We discussed algebraic relations in the space of form factors and outlined the set of problems related to the form factors of the descendant fields⁸. In particular, we stress the role of quantum equations of motion by showing that they appear naturally in the form factor prescription. On the other hand, we demonstrated the fact that the equations of motion can be a very powerful tool in studying the ultraviolet asymptotics, within the conformal perturbation theory. Their application allows us to find coefficients in the short-distance asymptotics, which are rather difficult to study by direct methods. Another useful method, which we applied in the analysis of form factors and conformal perturbation theory, is the W -extended symmetry of the model. The W symmetric models of CFT and their integrable perturbations are, in general, rather complicated and we found that it is interesting that correlation functions can be effectively studied for such models, at least, in the particular cases. We also stress the necessity of a more deeper understanding of the fields, which satisfy the resonance condition. We have shown that these fields have unusual properties in both the form factor approach and the conformal perturbation theory, and have to be analyzed carefully.

In this paper, we did not consider ultraviolet expansions for correlation functions of ‘heavy’ fields O_a , which are in resonance with some field O_b (3)–(4). If we use the proposed form factor regularization prescription for these fields, then the calculation of the short-distance behavior for correlators becomes a more subtle problem, and terms, including $\log^2 r$, will appear in the short-distance expansion. The well-known example for this property is the correlation function $\langle \psi \bar{\psi}(x) \psi \bar{\psi}(0) \rangle$ in the Ising model.

Another interesting phenomenon in the study of resonance fields is that sometimes, the renormalization of primary fields is finite and does not require introducing the parameter ϵ for the correct definition of such deformed primary fields off-criticality. We propose to describe these problems in other publication.

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Appendix A. Short-distance expansion for order and disorder fields

For completeness, we also collect here the results [7] about the following correlation functions:

$$G_+(x) = \langle \sigma_1(x) \sigma_1^+(0) \rangle, \quad G_-(x) = \langle \mu_1(x) \mu_1^+(0) \rangle \quad (\text{A.1})$$

of order and disorder fields in scaling the Z_N Ising model. We considered the leading terms in the short-distance expansion:

$$G_{\pm}(x) = r^{-4d_I} (C_{\pm}^I + C_{\pm}^{\epsilon_1} \langle \epsilon_1 \rangle + C_{\pm}^{\epsilon_2} \langle \epsilon_2 \rangle + C_{\pm}^{E_1} \langle E_1 \rangle + \dots). \quad (\text{A.2})$$

⁸ Similar problems were independently discussed in paper [43]. See also [41, 42].

Up to the first-order perturbation theory, the structure functions entering this expression are

$$\begin{aligned}
 C_{\pm}^I &= 1 \pm \frac{\lambda\pi r^{2(1-2u)}(\gamma(u)\gamma(3u))^{1/2}\gamma(4u)}{2(1-4u)^2\gamma^3(2u)}, \\
 C_{\pm}^{\varepsilon_1} &= \mp \frac{r^{4u}(\gamma(u)\gamma(3u))^{1/2}}{2\gamma(2u)} + \frac{\lambda\pi r^2 u^2 \gamma(4u)\gamma^4(u)}{4\gamma^4(2u)}, \\
 C_{\pm}^{\varepsilon_2} &= \mp r^{12u} \frac{\lambda\pi r^{2(1-2u)} u^2 \gamma^3(u)(\gamma(5u))^{1/2}}{12(1+2u)^2\gamma^2(2u)(\gamma(3u))^{1/2}}, \\
 C_{\pm}^{E_1} &= -r^{2+4u} \frac{u(1-4u)(\gamma(u)\gamma(3u))^{1/2}}{2(1+2u)(1-2u)\gamma(2u)}.
 \end{aligned}
 \tag{A.3}$$

Note that the Ising model correlation function short-distance expansions [15, 39] agree with the $N \rightarrow 2$ expressions of these correlators as well⁹.

Appendix B. Two-particle form factors

Let the contour C go from infinity above the real axe, then around zero and then to infinity, below the real axe. Introduce the notation

$$S_2(x) = \exp \frac{1}{2} \int_C \frac{dt}{2\pi it} \frac{\sinh(x-2\pi)t}{\sinh^2(\pi t)} \log(-t).
 \tag{B.1}$$

Then the functions $\zeta^{(\dagger)}$, appearing in equation (32), read as

$$\begin{aligned}
 \zeta(\beta) &= \frac{i \sinh\left(\frac{\beta}{2}\right)}{2 \sinh\left(\frac{\beta}{2} + \frac{i\pi}{N}\right) \sinh\left(\frac{\beta}{2} - \frac{i\pi}{N}\right)} \frac{S_2\left(i\beta + 2\pi + \frac{2\pi}{N}\right) S_2\left(-i\beta + \frac{2\pi}{N}\right)}{S_2^2\left(2\pi + \frac{2\pi}{N}\right)}, \\
 \zeta^\dagger(\beta) &= \frac{1}{\cosh \frac{\beta}{2}} \frac{S_2^2\left(2\pi + \frac{2\pi}{N}\right)}{S_2\left(i\beta + 3\pi + \frac{2\pi}{N}\right) S_2\left(-i\beta + \pi + \frac{2\pi}{N}\right)}.
 \end{aligned}
 \tag{B.2}$$

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⁹ The same phenomenon was observed by V Belavin in the case [40] of the minimal models of the CFT $\mathcal{M}_{2,2n+1}$ perturbed first energy field.

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